$|\psi(t)\rangle = c_g(t)|g\rangle + c_e(t)|e\rangle = e^{-i\hat{H}t/\hbar}|\psi(t=0)\rangle$

$c_g(t) =$

$c_e(t) =$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$\boxed{\begin{array}{l}\otimes \text{ exponential of a matrix} \\ \quad \text{Find } \cdot \text{ eigenvalues} \\ \qquad\quad\ \ \cdot \text{ eigenvector} \\ \text{Homework}\end{array}}$

Heisenberg Eq. $\quad \partial_t \hat{O} = -\frac{i}{\hbar}[\hat{H}, \hat{O}]$

$\hat{O}$: $\sigma^\dagger\sigma = |e\rangle\langle e|$: population operator
occupation

$H = \hbar\omega_\sigma \sigma^\dagger\sigma + \hbar\Omega(\sigma + \sigma^\dagger)$

$\partial_t \sigma^\dagger\sigma = -\frac{i}{\hbar}[H, \sigma^\dagger\sigma] \qquad [A,B] = AB - BA$

$H\sigma^\dagger\sigma = \hbar\omega_\sigma \sigma^\dagger\sigma\sigma^\dagger\sigma + \hbar\Omega(\sigma\sigma^\dagger\sigma + \sigma^\dagger\sigma^\dagger\sigma)$

$1 = \sigma^\dagger\sigma + \sigma\sigma^\dagger \qquad$ Normal ordering

$\sigma\sigma^\dagger = 1 - \sigma^\dagger\sigma$

$\sigma^\dagger\sigma\sigma^\dagger\sigma = \sigma^\dagger(1 - \sigma^\dagger\sigma)\sigma \qquad \sigma\sigma^\dagger\sigma = (1 - \sigma^\dagger\sigma)\sigma$

$= \sigma^\dagger\sigma - \sigma^{\dagger 2}\sigma^2 \qquad = \sigma - \sigma^\dagger\sigma^2$

$\sigma^{\dagger 2} = |e\rangle\langle g|e\rangle\langle g| = 0 \qquad \sigma = |g\rangle\langle e|$

$\sigma^2 = |g\rangle\langle e|g\rangle\langle e| = 0 \qquad \sigma^\dagger = |e\rangle\langle g|$

$\langle e|e\rangle = 1 \qquad \langle e|g\rangle = 0 \qquad \langle g|g\rangle = 1 \qquad \langle g|e\rangle = 0$

$|e\rangle\langle g|e\rangle\langle g| = |e\rangle\langle g|\ \langle g|e\rangle$

$|e\rangle\langle g|$: Tensor product $\quad \vec{a} \otimes \vec{b}$: tensor

$\langle g|e\rangle$: Inner product : $\vec{a}\cdot\vec{b}$: scalar

$|e\rangle\langle g|e\rangle\langle g| \qquad$ H.P

$H\sigma^\dagger\sigma = \hbar\omega_\sigma \sigma^\dagger\sigma + \hbar\Omega\sigma \qquad \partial_t \sigma^\dagger\sigma$

$\sigma^\dagger\sigma H =$

$(ABC\cdots Z)^\dagger = Z^\dagger \cdots C^\dagger B^\dagger A^\dagger$

$(H\sigma^\dagger\sigma)^\dagger = \sigma^\dagger(\sigma^\dagger)^\dagger H^\dagger$

$= \sigma^\dagger\sigma H$

$\sigma^\dagger\sigma H = \hbar\omega_\sigma \sigma^\dagger\sigma + \hbar\Omega\sigma^\dagger$

$[H, \sigma^\dagger\sigma] = \hbar\omega_\sigma\sigma^\dagger\sigma + \hbar\Omega\sigma - (\hbar\omega_\sigma\sigma^\dagger\sigma + \hbar\Omega\sigma^\dagger)$

$= \hbar\Omega(\sigma - \sigma^\dagger)$

$$\boxed{\partial_t\ \sigma^\dagger\sigma = \frac{-i}{\hbar}\hbar\Omega(\sigma-\sigma^\dagger) = -i\Omega(\sigma-\sigma^\dagger)}$$

$\langle\psi|\partial_t\sigma^\dagger\sigma|\psi\rangle = \partial_t\langle\psi|\sigma^\dagger\sigma|\psi\rangle$: Because $|\psi\rangle$ is time-independent

$= \partial_t\langle\sigma^\dagger\sigma\rangle$

$\partial_t\langle\psi|\sigma^\dagger\sigma|\psi\rangle = \langle\psi|\left(-i\Omega(\sigma-\sigma^\dagger)\right)|\psi\rangle$

$= -i\Omega\left(\langle\psi|\sigma|\psi\rangle - \langle\psi|\sigma^\dagger|\psi\rangle\right)$

$$\boxed{\partial_t \langle \sigma^\dagger \sigma \rangle = -i\Omega \left(\langle \sigma \rangle - \langle \sigma^\dagger \rangle \right)}$$

- Get $\partial_t \langle \sigma \rangle$ and $\partial_t \langle \sigma^\dagger \rangle$ → Homework!

→ $C_g(t)$ and $C_e(t)$ provide the same information as $\langle \sigma^\dagger \sigma \rangle(t)$ and $\langle \sigma \rangle(t)$

→ $\langle \sigma^\dagger \sigma \rangle(t) = \langle \psi(t) | \sigma^\dagger \sigma | \psi(t) \rangle$

$\sigma^\dagger \sigma = |e\rangle\langle g|g\rangle\langle e|$
$= |e\rangle\langle e|$

$|\psi(t)\rangle = C_g(t)|g\rangle + C_e(t)|e\rangle$

$\sigma^\dagger \sigma |\psi(t)\rangle = C_g(t)\, |e\rangle\langle e|g\rangle + C_e(t)\, |e\rangle\langle e|e\rangle$

$= C_e(t)\, |e\rangle$

$\langle \psi(t)| = C_g^*(t) \langle g| + C_e^*(t) \langle e|$

$\langle \psi(t)| \sigma^\dagger \sigma |\psi(t)\rangle = \left(C_g^*(t) \langle g| + C_e^*(t) \langle e| \right) C_e(t) |e\rangle$

$= |C_e(t)|^2$

$1 = |C_g(t)|^2 + |C_e(t)|^2 \Rightarrow |C_g(t)|^2 = 1 - \langle \sigma^\dagger \sigma \rangle(t)$

$C_g = \alpha e^{i\theta}$ $\quad \alpha, \theta \in \mathbb{R}$

$C_e = \beta e^{i\phi}$ $\quad \beta, \phi \in \mathbb{R}$

$\langle \sigma^\dagger \sigma \rangle = \beta^2$ $\quad \alpha^2 = 1 - \langle \sigma^\dagger \sigma \rangle$

$|\psi\rangle = \alpha e^{i\theta} |g\rangle + \beta e^{i\phi} |e\rangle = e^{i\theta} \left(\alpha |g\rangle + \beta e^{i(\phi - \theta)} |e\rangle \right)$

$\langle \psi | \hat{O} | \psi \rangle$

$$\langle\psi| = e^{-i\theta}\left(\alpha\langle g| + \beta e^{-i(\phi-\theta)}\langle e|\right) \qquad |\psi\rangle,\ |\psi\rangle e^{i\phi}$$

$$\Rightarrow \langle\sigma\rangle(t) = \langle\psi(t)|\sigma|\psi(t)\rangle$$

$$\sigma|\psi(t)\rangle = C_g(t)|g\rangle\langle e|g\rangle + C_e(t)|g\rangle\langle e|e\rangle \qquad \sigma = |g\rangle\langle e|$$

$$= C_e(t)|g\rangle$$

$$\langle\psi(t)|\sigma|\psi(t)\rangle = \left(C_g^*(t)\langle g| + C_e^*(t)\langle e|\right)C_e(t)|g\rangle$$

$$\langle\sigma\rangle = C_g^*(t)C_e(t)$$

$$\langle\sigma^+\rangle = C_g(t)C_e^*(t)$$

$$C_g^* = \alpha e^{-i\theta} \qquad C_g^* C_e = \alpha\beta e^{i(\phi-\theta)} = \langle\sigma\rangle$$

$$C_e = \beta e^{i\phi}$$

$$\alpha = \sqrt{1-\langle\sigma^+\sigma\rangle}$$

$$\beta = \sqrt{\langle\sigma^+\sigma\rangle}$$

$$e^{i(\phi-\theta)} = \frac{\langle\sigma\rangle}{\alpha\beta} = \frac{\langle\sigma\rangle}{\sqrt{1-\langle\sigma^+\sigma\rangle}\sqrt{\langle\sigma^+\sigma\rangle}}$$

Solve the diff. eqs. and get

$\langle\sigma^+\sigma\rangle(t) =$ $\qquad$ $0 \leq \langle\sigma^+\sigma\rangle(t) \leq 1$

$\langle\sigma\rangle(t) =$