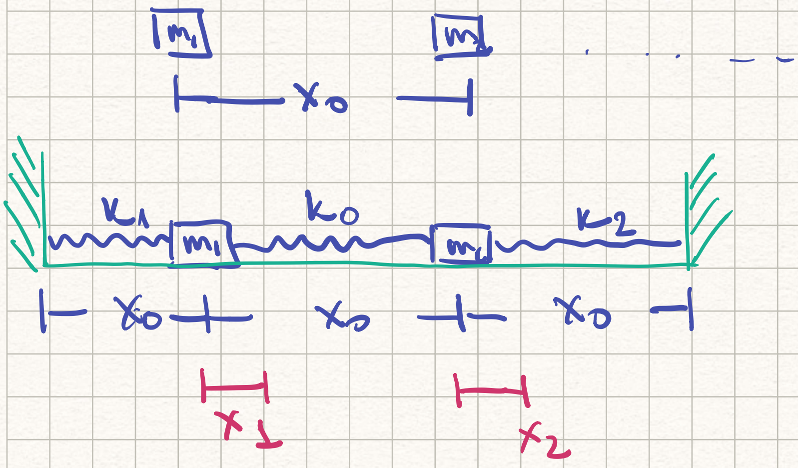
Coupled Harmonic oscillators



Chain of masses

Toy-model for a 1D crystal.

Newton's eq. of motion:

$$\boxed{\ddot{x}_j = \frac{\partial^2}{\partial t^2} x_j}$$

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_0 (x_2 - x_1)$$
$$m_2 \ddot{x}_2 = -k_2 x_2 - k_0 (x_2 - x_1)$$

Coupled. diff. Eqs.

Ansatz:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}$$

Harmonic solution

$$\ddot{x}_1 = -\omega^2 A_1 e^{i\omega t}$$
$$\ddot{x}_2 = -\omega^2 A_2 e^{i\omega t} \Rightarrow \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} = -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}$$

$$-m_1 \omega^2 A_1 e^{i\omega t} = -k_1 A_1 e^{i\omega t} + k_0 (A_2 - A_1) e^{i\omega t}$$
$$-m_2 \omega^2 A_2 e^{i\omega t} = -k_2 A_2 e^{i\omega t} - k_0 (A_2 - A_1) e^{i\omega t}$$

Algebraic eqs. for A_1 and A_2.

Divide by $e^{i\omega t}$ everywhere and reorganize

$$-m_1 \omega^2 A_1 = -k_1 A_1 + k_0 (A_2 - A_1)$$
$$0 = (m_1 \omega^2 - k_1) A_1 + k_0 (A_2 - A_1)$$

$$0 = (m_1\omega^2 - k_1 - k_0)A_1 + k_0 A_2$$
$$0 = k_0 A_1 + (m_2\omega^2 - k_2 + k_0)A_2$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_1\omega^2 - k_1 - k_0 & k_0 \\ k_0 & m_2\omega^2 - k_2 + k_0 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \qquad \vec{0} = M\vec{v}$$

This has a solution $\det(M) = 0$

$$0 = \det(M) = (m_1\omega^2 - k_1 - k_0)(m_2\omega^2 - k_2 + k_0) - k_0^2$$
$$0 = \omega^4 m_1 m_2 - \omega^2(k_0 m_1 + k_2 m_1 + k_0 m_2 + k_1 m_2) + k_0k_1 + k_0k_2 + k_1k_2$$

$$0 = x^2 a + bx + c \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \omega^2$$

$$\omega^2 = \frac{k_2 m_1 + k_1 m_2 + k_0(m_1 + m_2)}{2m_1m_2} \pm$$

Normal modes of oscillation

$$\pm \frac{\left\{\left[(k_0+k_1)m_1 + (k_0+k_1)m_2\right]^2 - 4m_1m_2(k_0k_1 + k_0k_2 + k_1k_2)\right\}^{1/2}}{2m_1m_2}$$

$\rightarrow$ Makes sense $k_1 = k_2 = k$

$m_1 = m_2 = m$

$$\omega^2 = \begin{cases} \dfrac{k}{m} \\[2ex] \dfrac{k + 2k_0}{m} \end{cases}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_1\omega^2 - k_1 - k_0 & k_0 \\ k_0 & m_2\omega^2 - k_2 + k_0 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$\omega^2 = k/m$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -k_0 & k_0 \\ k_0 & -k_0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

1st eq.

$0 = -k_0 A_1 + k_0 A_2$

$0 = -A_1 + A_2$

2nd eq.

$0 = k_0 A_1 - k_0 A_2$

$0 = A_1 - A_2$

$\boxed{A_1 = A_2}$

$$\omega^2 = \frac{k + 2k_0}{m}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k_0 & k_0 \\ k_0 & k_0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$0 = k_0 A_1 + k_0 A_2$

$0 = A_1 + A_2$

$\boxed{A_1 = -A_2}$

Every oscillation of these masses can be expressed as a superposition of the motion with the normal modes

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_1 t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_1 t}$$

$$+ c_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_2 t} + c_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega_2 t}$$

$\omega_1^2 = \frac{k}{m}$

$\omega_2^2 = \frac{k+2k_0}{m}$

$c_1 = c_2^* = B_1 e^{i\phi_1}/2 \qquad c_3 = c_4^* = B_2 e^{i\phi_2}/2$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = B_1 \cos(\omega_1 t + \phi_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B_2 \cos(\omega_2 t + \phi_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Particular cases

If $B_1 = 0$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = B_2 \cos(\omega_2 t + \phi_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ out of phase

If $B_2 = 0$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = B_1 \cos(\omega_1 t + \phi_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ they are synchronized

$\boxed{\text{Maths}}$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_1\omega^2 - K_1 - K_0 & K_0 \\ K_0 & m_2\omega^2 - K_2 + K_0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Factorize a minus sign, $m_1 = m_2 = m$ $K_1 = K_2 = K$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} K + K_0 - m\omega^2 & -K_0 \\ -K_0 & K - K_0 - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$0 = (K + K_0)A_1 \quad - K_0 A_2 \quad - m\omega^2 A_1$$

$$0 = -K_0 A_1 + (K - K_0) A_2 \quad - m\omega^2 A_2$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} K + K_0 & -K_0 \\ -K_0 & K - K_0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} - m\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\frac{1}{m} \begin{pmatrix} K + K_0 & -K_0 \\ -K_0 & K - K_0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

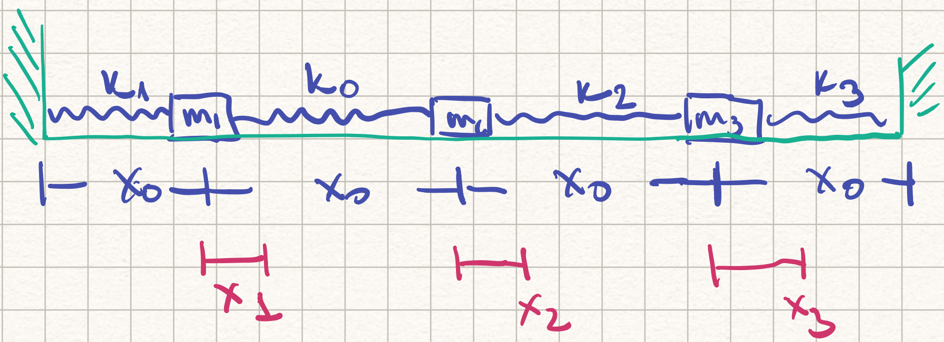
$$\mathbb{M} \vec{v} = \underline{\underline{\lambda}} \vec{v}$$ λ: eigenvalues of $\mathbb{M}$

ω^2 is the eigenvalue of matrix $\frac{1}{m} \begin{pmatrix} K + K_0 & -K_0 \\ -K_0 & K - K_0 \end{pmatrix}$ with eigenvector $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$

$\boxed{\text{3 masses}}$

$m_1 = m_2 = m_3 = m \qquad \omega_0^2 = \frac{k}{m}$

$k_1 = k_2 = k_3 = k_0 = k$



$$m_1 \ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m_2 \ddot{x}_2 = k(x_3 - x_2) - k(x_2 - x_1)$$
$$m_3 \ddot{x}_3 = -k(x_3 - x_2) - kx_3$$

Ansatz

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} e^{i\omega t}$$

Using the Ansatz:

$$-\omega^2 m A_1 = -kA_1 + k(A_2 - A_1)$$
$$-\omega^2 m A_2 = k(A_3 - A_2) - k(A_2 - A_1)$$
$$-\omega^2 m A_3 = -k(A_3 - A_2) - kA_3$$

$\frac{k}{m} = \omega_0^2$

$$-\omega^2 A_1 = -\omega_0^2 A_1 + \omega_0^2 (A_2 - A_1) = -2\omega_0^2 A_1 + \omega_0^2 A_2$$
$$-\omega^2 A_2 = \omega_0^2 (A_3 - A_2) - \omega_0^2 (A_2 - A_1) = \omega_0^2 A_1 - 2\omega_0^2 A_2 + \omega_0^2 A_3$$
$$-\omega^2 A_3 = -\omega_0^2 (A_3 - A_2) - \omega_0^2 A_3 = \omega_0^2 A_2 - 2\omega_0^2 A_3$$

$$\begin{pmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & \omega - 2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & \omega - 2\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \alpha & \beta \\ 0 & \beta & \alpha \end{pmatrix} = \alpha(\alpha^2 - 2\beta^2) = 0$$

$\alpha = 0$

$\alpha^2 = 2\beta^2$

$\alpha = \omega^2 - 2\omega_0^2$

$\beta = \omega_0^2$

$\omega^2 - 2\omega_0^2 = 0$

$\boxed{\omega^2 = 2\omega_0^2}$

$(\omega^2 - 2\omega_0^2)^2 = 2\omega_0^4$

$\omega^4 + 4\omega_0^4 - 4\omega^2\omega_0^2 = 2\omega_0^4$

$\omega^4 + 2\omega_0^4 - 4\omega^2\omega_0^2 = 0$

$$\omega^2 = \frac{4\omega_0^2 \pm \sqrt{16\omega_0^4 - 8\omega_0^4}}{2}$$

$$= \frac{4\omega_0^2 \pm \sqrt{8\omega_0^4}}{2}$$

$$= \frac{4\omega_0^2 \pm 2\sqrt{2}\,\omega_0^2}{2}$$

$$\boxed{\omega^2 = (2 \pm \sqrt{2})\omega_0^2}$$

3 masses → 3 normal modes

N masses → N normal modes

$\boxed{\omega^2 = 2\omega_0^2}$

$$\begin{pmatrix} 0 & \omega_0^2 & 0 \\ \omega_0^2 & 0 & \omega_0^2 \\ 0 & \omega_0^2 & 0 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\omega_0^2 A_2 = 0 \Rightarrow A_2 = 0$

$\omega_0^2 (A_1 + A_3) = 0 \Rightarrow A_1 = -A_3$

$$A\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\omega^2 = (2 \pm \sqrt{2})\omega_0^2$

$$\omega_0^2 \begin{pmatrix} \pm\sqrt{2} & 1 & 0 \\ 1 & \pm\sqrt{2} & 1 \\ 0 & 1 & \pm\sqrt{2} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A\begin{pmatrix} 1 \\ \mp\sqrt{2} \\ 1 \end{pmatrix}$$

$\omega_s^2 = (2 - \sqrt{2})\omega_0^2$ $\omega_m^2 = 2\omega_0^2$ $\omega_f^2 = (2 + \sqrt{2})\omega_0^2$

A_s, ϕ_s A_m, ϕ_m A_f, ϕ_f

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}(t) = A_m \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos\left(\overbrace{\sqrt{2}\,\omega_0}^{\omega_m} t + \phi_m\right)$$

$$+ A_f \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \cos\left(\overbrace{\sqrt{2+\sqrt{2}}\,\omega_0}^{\omega_f} t + \phi_f\right)$$

$$+ A_s \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos\left(\overbrace{\sqrt{2-\sqrt{2}}\,\omega_0}^{\omega_s} t + \phi_s\right)$$